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Noether's theorem and Lie symmetries for time-dependent Hamilton-Lagrange systems

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Noether and Lie symmetry analyses based on point transformations that depend on time and spatial coordinates will be reviewed for a general class of time-dependent Hamiltonian systems. The resulting symmetries are expressed in the form of generators whose time-dependent coefficients follow as solutions of sets of ordinary differential ("auxiliary") equations. The interrelation between the Noether and Lie sets of auxiliary equations will be elucidated. The auxiliary equations of the Noether approach will be shown to admit invariants for a much broader class of potentials, compared to earlier studies. As an example, we work out the Noether and Lie symmetries for the time-dependent Kepler system. The Runge-Lenz vector of the time-independent Kepler system will be shown to emerge as a Noether invariant if we adequately interpret the pertaining auxiliary equation. Furthermore, additional nonlocal invariants and symmetries of the Kepler system will be isolated by identifying further solutions of the auxiliary equations that depend on the explicitly known solution path of the equations of motion. Showing that the invariants remain unchanged under the action of different symmetry operators, we demonstrate that a unique correlation between a symmetry transformation and an invariant does not exist.

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I. INTRODUCTION

Analytical approaches aiming to analyze the particular properties of a given dynamical system may successfully take advantage of the formalism of infinitesimal symmetry transformations that have been worked out by Lie [1] and Noether [2]. In this paper, we will review these approaches. Specifically, both Noether and Lie symmetries will be worked out on the basis of point transformations with variations depending on time and spatial coordinates for a general class of explicitly time-dependent Hamiltonian systems. This parallel treatment will enable us to compare these symmetry analyses, and to unveil both their close relationship and their differences. We will furthermore contribute to the ongoing discussion on how these symmetries are related to the invariants of a given dynamical system.

The results of the symmetry analyses are obtained in the form of generators of symmetry transformations. The particular form of these generators is constituted by time-dependent coefficients that are given as solutions of ordinary differential ("auxiliary") equations. In order to obtain the full "spectrum" of these solutions, the auxiliary equations *and* the system's equations of motion must be conceived as a coupled set [3–5]. The particular solutions of the auxiliary equations that decouple from the solutions of the equations of motion can then be seen to yield the generators of the "fundamental" system symmetries.

As an example, we work out the Noether and Lie symmetry analyses for the time-dependent Kepler system. The specific auxiliary equations are directly obtained from the general formulation derived beforehand. All known invariants and Lie symmetries will be shown to emerge from the solutions of these auxiliary equations. It is shown in particular that the Runge-Lenz vector of the time-independent Kepler system is obtained as a classical Noether invariant—hence an invariant that arises from a point transformation that depends on the vector of spatial coordinates $\vec{q}$ and time t—if we interpret the pertaining auxiliary equation appropriately. Furthermore, not previously reported nonlocal Noether invariants and Lie symmetries of the Kepler system will be isolated working out additional solutions of the respective auxiliary equations.

The variation of the Noether invariants will be shown to vanish under different Lie and Noether symmetry transformations. We thereby demonstrate that a unique correlation between a symmetry and a related invariant does not exist.

We start our analysis with a review of infinitesimal point transformation and their generators in space-time. This will be particularly helpful to clarify our notation and to render our paper as self-contained as possible. In this context, the brief presentation of Noether's theorem will largely facilitate the understanding of the Noether symmetry analysis of the general Hamiltonian system that is governed by the potential $V(\vec{q},t)$, as well as the subsequent Lie symmetry analysis.

II. INFINITESIMAL POINT TRANSFORMATIONS

Given a classical n-degree-of-freedom dynamical system of particles, an infinitesimal point transformation denotes a transformation that maps "points" in configuration space and time into infinitesimal neighboring "points": $(\vec{q},t)\mapsto(\vec{q}\,',t')$, the primes indicating the transformed quantities. Formally, such a point transformation in the $(\vec{q},t)$ space-time may be defined in terms of an infinitesimal parameter ε by

$$t' = t + \delta t, \quad \delta t = \varepsilon\,\xi(\vec{q},t), \tag{1a}$$

$$q_i' = q_i + \delta q_i, \quad \delta q_i = \varepsilon\,\eta_i(\vec{q},t). \tag{1b}$$

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In order to derive the transformation rules for $\dot{q}_i$ and $\ddot{q}_i$ for the infinitesimal point transformation defined by Eqs. (1a) and (1b), we must be aware that the coordinates q_i and the time t are transformed *simultaneously*. The quantity $\delta\dot{q}_i$ follows from the consideration that $\dot{q}_i'$ is given by the derivative of the transformed coordinate q_i' with respect to the transformed time t'. According to the transformation rules (1a) and (1b), we thus find [6]

$$\dot{q}_i' = \frac{dq_i'}{dt'} = \frac{dq_i + \varepsilon d\eta_i}{dt + \varepsilon d\xi} = \frac{\dot{q}_i + \varepsilon\dot{\eta}_i}{1+\varepsilon\dot{\xi}} = \dot{q}_i + \varepsilon(\dot{\eta}_i - \dot{\xi}\dot{q}_i) + O(\varepsilon^2),$$

which means that the first-order variation $\delta\dot{q}_i$ is given by

$$\delta\dot{q}_i = \varepsilon[\dot{\eta}_i(\vec{q},t) - \dot{\xi}(\vec{q},t)\dot{q}_i]. \tag{1c}$$

The infinitesimal point transformation (1a) and (1b) thus uniquely determines the mapping of the $\dot{q}_i'$. Similarly, we find the transformation rule for the $\ddot{q}_i'$ from

$$\ddot{q}_i' = \frac{d\dot{q}_i'}{dt'} = \frac{\ddot{q}_i + \varepsilon(\ddot{\eta}_i - \dot{\xi}\ddot{q}_i - \ddot{\xi}\dot{q}_i)}{1+\varepsilon\dot{\xi}} = \ddot{q}_i + \varepsilon(\ddot{\eta}_i - 2\dot{\xi}\ddot{q}_i - \ddot{\xi}\dot{q}_i) + O(\varepsilon^2),$$

which yields the variation $\delta\ddot{q}_i$ to first order in ε,

$$\delta\ddot{q}_i = \varepsilon(\ddot{\eta}_i - 2\dot{\xi}\ddot{q}_i - \ddot{\xi}\dot{q}_i). \tag{1d}$$

Given an arbitrary analytic function $u(\vec{q},t)$ of the n-dimensional vector of particle positions and time, the function's variation $\delta u = u(\vec{q}\,',t') - u(\vec{q},t)$ that is induced by virtue of the point transformation (1) is given by

$$\delta u = \frac{\partial u}{\partial t}\delta t + \sum_{i=1}^{n} \frac{\partial u}{\partial q_i}\delta q_i = \varepsilon \boldsymbol{U} u(\vec{q},t),$$

the operator $\boldsymbol{U}$ denoting the generator of the infinitesimal point transformation (1),

$$\boldsymbol{U} = \xi(\vec{q},t)\frac{\partial}{\partial t} + \sum_{i=1}^{n} \eta_i(\vec{q},t)\frac{\partial}{\partial q_i}. \tag{2}$$

The variation $\delta v = v(\vec{q}\,',\dot{\vec{q}}\,',t') - v(\vec{q},\dot{\vec{q}},t)$ of an arbitrary analytic function $v(\vec{q},\dot{\vec{q}},t)$ follows as:

$$\delta v = \frac{\partial v}{\partial t}\delta t + \sum_{i=1}^{n}\left(\frac{\partial v}{\partial q_i}\delta q_i + \frac{\partial v}{\partial \dot{q}_i}\delta\dot{q}_i\right) = \varepsilon \boldsymbol{U}' v(\vec{q},\dot{\vec{q}},t),$$

which means that the first "extension" $\boldsymbol{U}'$ of the generator (2) is given by

$$\boldsymbol{U}' = \boldsymbol{U} + \sum_{i=1}^{n} \eta_i' \frac{\partial}{\partial \dot{q}_i}, \qquad \eta_i' = \dot{\eta}_i - \dot{\xi}\dot{q}_i. \tag{3}$$

Finally, the variation $\delta w = w(\vec{q}\,',\dot{\vec{q}}\,',\ddot{\vec{q}}\,',t') - w(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)$ of an arbitrary analytic function $w(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t)$ is obtained as

$$\delta w = \varepsilon \boldsymbol{U}'' w(\vec{q},\dot{\vec{q}},\ddot{\vec{q}},t),$$

with $\boldsymbol{U}''$ the second "extension" of the generator (2),

$$\boldsymbol{U}'' = \boldsymbol{U}' + \sum_{i=1}^{n} \eta_i'' \frac{\partial}{\partial \ddot{q}_i},$$

$$\eta_i'' = \ddot{\eta}_i - 2\dot{\xi}\ddot{q}_i - \ddot{\xi}\dot{q}_i = \frac{d}{dt}\eta_i' - \dot{\xi}\ddot{q}_i. \tag{4}$$

We will make use of the second extension $\boldsymbol{U}''$ of the generator $\boldsymbol{U}$ in Sec. V B for a Lie symmetry analysis of a general time-dependent Lagrangian system. Beforehand, the first extension $\boldsymbol{U}'$ will be needed in our review of Noether's theorem to be presented in the following section.

III. REVIEW OF NOETHER'S THEOREM

Noether's theorem [2,7,8] relates the conserved quantities of an n-degree-of-freedom Lagrangian system $L(\vec{q},\dot{\vec{q}},t)$ to infinitesimal point transformations (1) that leave the Lagrange action $L dt$ invariant. We now work out this theorem in the special form that emerges from the infinitesimal point transformation (1). Among the general set of point transformations defined by Eq. (1), we consider exactly those that leave the action $L dt$ for a given Lagrangian $L(\vec{q},\dot{\vec{q}},t)$ invariant,

$$L(\vec{q},\dot{\vec{q}},t)dt \overset{!}{=} L'(\vec{q}\,',\dot{\vec{q}}\,',t')dt'. \tag{5}$$

Note that we allow the Lagrangian itself to change its functional form by virtue of the point transformation in order to satisfy the condition (5). As the system's equations of motion follow directly from the variation of the action integral by virtue of Hamilton's principle $\delta\int L dt = 0$, the condition (5) implies the requirement that the particular symmetry transformation (1) must sustain the form of the equations of motion. This means that the point transformation (1) maps the action integral into another representation of the *same* action integral. In other words, we do *not* transform a physical system into a different one, but regard a given Lagrangian system $L(\vec{q},\dot{\vec{q}},t)$ from an infinitesimally dislodged "viewpoint" in order to isolate its inherent symmetries.

The *functional* relation between L' and L may be expressed introducing a gauge function $f(\vec{q},t)$,

$$L'(\vec{q}\,',\dot{\vec{q}}\,',t') = L + \delta L + \cdots = L(\vec{q}\,',\dot{\vec{q}}\,',t') - \varepsilon\frac{df}{dt} + O(\varepsilon^2). \tag{6}$$

For the relation (6) to hold in general, it is necessary and sufficient [7] that $f(\vec{q},t)$ depend on $\vec{q}$ and t only since, according to Eq. (1c), the transformation $\dot{\vec{q}}\mapsto\dot{\vec{q}}'$ is uniquely determined by $\vec{q}\mapsto\vec{q}'$ and $t\mapsto t'$. Inserting Eq. (6) into the condition for the invariant Lagrange action (5), we get to first order in ε,

$$L(\vec{q}',\dot{\vec{q}}',t')dt' = L(\vec{q},\dot{\vec{q}},t)dt + \varepsilon\frac{df(\vec{q},t)}{dt}dt. \tag{7}$$

On the other hand, the connection between $L(\vec{q}',\dot{\vec{q}}',t')$ and $L(\vec{q},\dot{\vec{q}},t)$ is determined by the "extended" operator $\boldsymbol{U}'$ of Eq. (3),

$$L(\vec{q}',\dot{\vec{q}}',t') = L(\vec{q},\dot{\vec{q}},t) + \varepsilon\boldsymbol{U}'L(\vec{q},\dot{\vec{q}},t).$$

To first order in ε, Eq. (7) thus yields the auxiliary equation for $f(\vec{q},t)$, replacing dt' according to $dt'=(1+\varepsilon\dot{\xi})dt$,

$$\frac{df(\vec{q},t)}{dt} = \dot{\xi}L + \boldsymbol{U}'L. \tag{8}$$

With the operators $\boldsymbol{U}$ and $\boldsymbol{U}'$, given by Eqs. (2) and (3), respectively, the explicit form of Eq. (8) reads

$$\frac{df(\vec{q},t)}{dt} = \dot{\xi}L + \xi\frac{\partial L}{\partial t} + \sum_{i=1}^{n}\left(\eta_i\frac{\partial L}{\partial q_i} + (\dot{\eta}_i - \dot{q}_i\dot{\xi})\frac{\partial L}{\partial \dot{q}_i}\right). \tag{9}$$

We may conceive Eq. (9) as a condition for the yet unspecified functions $\xi(\vec{q},t)$ and $\eta_i(\vec{q},t)$. Only those point transformations (1) whose constituents ξ and η_i satisfy Eq. (9) maintain the Lagrange action Ldt for the given Lagrangian $L(\vec{q},\dot{\vec{q}},t)$.

The terms of Eq. (9) can directly be split into a total time derivative and a sum containing the Euler-Lagrange equations of motion,

$$\frac{d}{dt}\left[f(\vec{q},t) - \xi L + \sum_{i=1}^{n}(\xi\dot{q}_i - \eta_i)\frac{\partial L}{\partial\dot{q}_i}\right] + \sum_{i=1}^{n}(\xi\dot{q}_i - \eta_i)\times\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}\right) = 0. \tag{10}$$

Along the system trajectory $(\vec{q}(t),\dot{\vec{q}}(t))$ given by the solutions of the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i} = 0, \quad i=1,\ldots,n, \tag{11}$$

the related terms in Eq. (10) vanish. This means that the time integral I of the remaining terms

$$I = \sum_{i=1}^{n}(\xi\dot{q}_i - \eta_i)\frac{\partial L}{\partial\dot{q}_i} - \xi L + f(\vec{q},t) \tag{12}$$

constitutes a conserved quantity, i.e., a constant of motion for the Lagrange system $L(\vec{q},\dot{\vec{q}},t)$. The invariant given by Eq. (12) together with the differential equation (9) for $f(\vec{q},t)$ is commonly referred to as Noether's theorem. Starting from the initial condition $(\vec{q}(t_0),\dot{\vec{q}}(t_0))$, the system's state $(\vec{q}(t),\dot{\vec{q}}(t))$ is uniquely determined by the equations of motion (11), which in turn follow from Hamilton's principle $\delta\int L dt = 0$. Writing the variation $\delta\int L' dt' = 0$ of the infinitesimally transformed system in terms of the original coordinates, we obtain *in addition* to the equations of motion (11) the quantity I that is conserved by virtue of the symmetry transformation (1). Thus, the requirement $\delta\int L' dt' = 0$ may be seen as a generalization of Hamilton's principle that yields both the equations of motion *and* a phase-space symmetry relation embodied in the invariant I. In general, Eq. (9) for $f(\vec{q},t)$ depends on $\dot{\vec{q}}(t)$, hence on the solutions of the equations of motion (11).

Equation (10) exposes that the Noether invariant (12) emerges simultaneously with the time evolution of the system trajectory as the solution of the Euler-Lagrange equations (11). Alternatively, Noether's theorem can be interpreted as a coupled set of differential equations with the time t the common independent variable. This coupled set consists of both the Euler-Lagrange equations of motion (11) and an additional conditional equation for $f(\vec{q},t)$. In this regard, it can be considered as a generalized Ermakov [9] system whose time-dependent solutions form together the invariant of Eq. (12).

IV. NOETHER'S THEOREM IN HAMILTONIAN DESCRIPTION

From the definition of the Legendre transformation

$$L(\vec{q},\dot{\vec{q}},t) = \sum_{i=1}^{n} p_i\dot{q}_i - H(\vec{q},\vec{p},t) \tag{13}$$

that maps a given Lagrangian $L(\vec{q},\dot{\vec{q}},t)$ into the corresponding Hamiltonian $H(\vec{q},\vec{p},t)$, one finds the relations

$$p_i = \frac{\partial L}{\partial\dot{q}_i}, \quad \dot{p}_i = \frac{\partial L}{\partial q_i},$$

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} = -\frac{dH}{dt}. \tag{14}$$

Applying these transformation rules for the transition from a Lagrangian description of a dynamical system to a Hamiltonian description to the Noether invariant of Eq. (12), one immediately gets

$$I = \sum_{i}^{n}(\xi\dot{q}_i - \eta_i)p_i - \xi\sum_{i=1}^{n}p_i\dot{q}_i + \xi H + f(\vec{q},t),$$

which simplifies to the Hamiltonian formulation of Noether's theorem,

$$I=\xi(\vec{q},t)H(\vec{q},\vec{p},t)-\sum_{i=1}^{n}\eta_i(\vec{q},t)p_i+f(\vec{q},t). \tag{15}$$

The conditional equation for $f(\vec{q},t)$, given by Eq. (9), translates according to Eqs. (14),

$$\frac{df(\vec{q},t)}{dt}=-\xi\frac{dH}{dt}-\dot{\xi}H+\sum_{i=1}^{n}[\dot{\xi}p_i\dot{q}_i+\eta_i\dot{p}_i+\dot{\eta}_ip_i-\dot{\xi}p_i\dot{q}_i],$$

which can be written in the form of a total time derivative,

$$\frac{d}{dt}\left[\xi(\vec{q},t)H(\vec{q},\vec{p},t)-\sum_{i=1}^{n}\eta_i(\vec{q},t)p_i+f(\vec{q},t)\right]=0. \tag{16}$$

In the Hamiltonian formulation, the conditional equation (9) thus appears as the requirement that the total time derivative of the invariant (15) vanishes,

$$\frac{dI}{dt}\stackrel{!}{=}0. \tag{17}$$

For a Hamiltonian H with at most quadratic momentum dependence, the form (15) of the Noether invariant is compatible with an ansatz function consisting of quadratic and linear terms in the canonical momentum that has been used earlier by Lewis and Leach [10]. We thereby observe that this approach to work out an invariant is mathematically equivalent to a strategy based on Noether's theorem for this class of Hamiltonian systems.

V. HAMILTONIAN SYSTEM WITH A GENERAL TIME-DEPENDENT POTENTIAL

A. Noether symmetry analysis

To illustrate a particular Noether symmetry analysis, we consider the n-degree-of-freedom system of particles moving in an explicitly time-dependent potential $V(\vec{q},t)$,

$$H(\vec{p},\vec{q},t)=\sum_{i=1}^{n}\tfrac{1}{2}p_i^2+V(\vec{q},t). \tag{18}$$

The canonical equations following from Eq. (18) are

$$\dot{q}_i=\frac{\partial H}{\partial p_i}=p_i, \quad \dot{p}_i=-\frac{\partial H}{\partial q_i}=-\frac{\partial V}{\partial q_i}. \tag{19}$$

In the following, we work out the particular invariant I of the Hamiltonian system (18) that specializes the general Noether invariant in the form of Eq. (15). We hereby define a point mapping that is consistent with the Noether symmetry transformation (1). For the particular Hamiltonian (18), the general condition for $dI/dt=0$ of Eq. (16) reads

$$\frac{d}{dt}\left[\xi(\vec{q},t)\left(\sum_{i=1}^{n}\tfrac{1}{2}p_i^2+V(\vec{q},t)\right)-\sum_{i=1}^{n}\eta_i(\vec{q},t)p_i+f(\vec{q},t)\right]=0.$$

Inserting the canonical equations (19), the emerging equation can only be fulfilled globally for any particular vector of canonical momenta $\vec{p}$ if the sets of cubic, quadratic, and linear momentum terms vanish separately—and correspondingly the remaining terms that do not depend on the p_i,

$$\sum_i\sum_j\tfrac{1}{2}p_i^2p_j\frac{\partial\xi}{\partial q_j}=0, \tag{20a}$$

$$\sum_i\sum_j p_ip_j\left(\tfrac{1}{2}\delta_{ij}\frac{\partial\xi}{\partial t}-\frac{\partial\eta_i}{\partial q_j}\right)=0, \tag{20b}$$

$$\sum_i p_i\left(\frac{\partial f}{\partial q_i}-\frac{\partial\eta_i}{\partial t}+V\frac{\partial\xi}{\partial q_i}\right)=0, \tag{20c}$$

$$\sum_i\eta_i\frac{\partial V}{\partial q_i}+\frac{\partial\xi}{\partial t}V+\xi\frac{\partial V}{\partial t}+\frac{\partial f}{\partial t}=0. \tag{20d}$$

The notation δ_{ij} in Eq. (20b) stands for the Kronecker symbol. Since only a single momentum term appears in the sum of Eq. (20a), we may immediately conclude that the associated coefficient vanishes,

$$\frac{\partial\xi(\vec{q},t)}{\partial q_j}=0, \quad j=1,\ldots,n,$$

hence that $\xi(\vec{q},t)\equiv\beta(t)$ must be a function of t only. The double sum in Eq. (20b) vanishes globally for any $\vec{p}$ if $\partial\eta_i/\partial q_j$ cancels the $\dot{\xi}$ term up to a constant element a_{ij} of an antisymmetric matrix (a_{ij}),

$$\frac{\partial\eta_i(\vec{q},t)}{\partial q_j}=\tfrac{1}{2}\delta_{ij}\dot{\beta}(t)+a_{ij}, \quad a_{ij}=-a_{ji}.$$

In general form, the function $\eta_i(\vec{q},t)$ is thus given by

$$\eta_i(\vec{q},t)=\tfrac{1}{2}\dot{\beta}(t)q_i+\psi_i(t)+\sum_{j=1}^{n}a_{ij}q_j. \tag{21}$$

Herein, the $\psi_i(t)$ denote arbitrary functions of time only. The linear momentum terms of Eq. (20c) now require that

$$\frac{\partial f}{\partial q_i}=\frac{\partial\eta_i}{\partial t}.$$

Inserting the partial time derivative of Eq. (21), we find

$$\frac{\partial f(\vec{q},t)}{\partial q_i}=\tfrac{1}{2}\ddot{\beta}(t)q_i+\dot{\psi}_i(t).$$

This partial differential equation, too, may be generally integrated to yield

$$f(\vec{q},t)=\ddot{\beta}(t)\sum_{i=1}^{n}\tfrac{1}{4}q_i^2+\sum_{i=1}^{n}\dot{\psi}_i(t)q_i. \tag{22}$$

Now that η_i and f are specified by Eqs. (21) and (22), respectively, the invariant I of Eq. (15) can be expressed in terms of the yet unknown constants a_{ij} and functions of time $\beta(t)$ and $\psi_i(t)$,

$$I=\beta(t)H-\dot{\beta}(t)\sum_{i=1}^{n}\tfrac{1}{2}q_i p_i+\ddot{\beta}(t)\sum_{i=1}^{n}\tfrac{1}{4}q_i^2+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}q_i p_j+\sum_{i=1}^{n}(\dot{\psi}_i q_i-\psi_i p_i). \tag{23}$$

The functions $\beta(t)$ and $\psi_i(t)$ and the a_{ij} are determined from condition (20d), induced by the terms not depending on the canonical momenta p_i,

$$\dot{\beta}V+\beta\frac{\partial V}{\partial t}+\frac{\partial f}{\partial t}+\sum_{i=1}^{n}\eta_i\frac{\partial V}{\partial q_i}=0. \tag{24}$$

To obtain the explicit form of Eq. (24), we must insert Eq. (21) and the partial time derivative of Eq. (22). For potentials $V(\vec{q},t)$ that are not linear in $\vec{q}$, the $\psi_i(t)$ terms are the only ones that depend linearly on the canonical coordinates. Consequently, the sum of these terms must vanish separately. This means that two distinct differential equations are obtained, namely for those terms that do *not* depend on $\psi_i(t)$ and the remaining terms that depend on $\psi_i(t)$. The first group of terms of Eq. (24) form the following inhomogeneous linear differential equation for $\beta(t)$, keeping in mind that $a_{ij}=-a_{ji}$:

$$\dddot{\beta}(t)\sum_{i=1}^{n}\tfrac{1}{4}q_i^2+\dot{\beta}(t)\left[V(\vec{q}(t),t)+\sum_{i=1}^{n}\tfrac{1}{2}q_i\frac{\partial V}{\partial q_i}\right]+\beta(t)\frac{\partial V}{\partial t}+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}q_j\frac{\partial V}{\partial q_i}=0. \tag{25}$$

With $V(\vec{q}(t),t)$ the potential of Eq. (18), Eq. (25) represents an *ordinary* third-order differential equation along the solution path $\vec{q}(t)$ of the canonical equations (19) with the time t the independent variable. The general solution of Eq. (25) is given by the linear combination of its homogeneous part, together with a particular solution of the inhomogeneous equation. According to the existence and uniqueness theorem for linear ordinary differential equations, a unique solution $\beta(t)$ of the initial value problem (25) exists as long as its coefficients $\vec{q}(t)$, $V(\vec{q}(t),t)$, and its partial derivatives are continuous along the independent variable t. Otherwise, the function $\beta(t)$ may cease to exist at some finite instant of time t_1, which means that the related invariant exists within the limited time span $t_0\leqslant t<t_1$ only.

With our understanding of the auxiliary equation (25) as an ordinary differential equation along the *known* trajectory $\vec{q}(t)$, we differ from earlier studies of Lewis and Leach [10]. These authors conceived the auxiliary equation as a partial differential equation for potentials $V(\vec{q},t)$. Only those potentials that constitute a general solution of Eq. (25) were depicted to admit an invariant I. We observe here that the invariant I of Eq. (23) exists as well for the far more general class of potentials $V(\vec{q}(t),t)$ that admit a solution $\beta(t)$ of Eq. (25) along the trajectory $\vec{q}(t)$.

For the functions $\psi_i(t)$, Eq. (24) yields the condition

$$\sum_{i=1}^{n}\left(\ddot{\psi}_i(t)q_i+\psi_i(t)\frac{\partial V}{\partial q_i}\right)=0. \tag{26}$$

With $\psi_i(t)$ satisfying Eq. (26), the ψ_i-dependent terms of Eq. (23) form the separate invariant

$$I_\psi=\sum_{i=1}^{n}[\dot{\psi}_i(t)q_i-\psi_i(t)\,p_i]. \tag{27}$$

We recall that the functions $\psi_i(t)$ emerge in Eq. (21) as separate integration "constants" for each index $i=1,\dots,n$. Consequently, the invariant (27) and the related auxiliary equation (26) can be split into a set of n equations, respectively,

$$\ddot{\psi}_i(t)q_i-\psi_i(t)\dot{p}_i=0,\quad i=1,\dots,n, \tag{28}$$

$$I_{\psi_i}=\dot{\psi}_i(t)q_i-\psi_i(t)p_i,\quad i=1,\dots,n, \tag{29}$$

which means that the invariant I can be written as a sum of invariants $I=I_\beta+\Sigma_i I_{\psi_i}$. The $\psi_i(t)$-independent terms of Eq. (23) thus form the invariant I_β, which reads, inserting the Hamiltonian (18),

$$I_\beta=\beta(t)\left[\sum_{i=1}^{n}\tfrac{1}{2}p_i^2+V(\vec{q}(t),t)\right]-\dot{\beta}(t)\sum_{i=1}^{n}\tfrac{1}{2}q_i p_i+\ddot{\beta}(t)\sum_{i=1}^{n}\tfrac{1}{4}q_i^2+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}q_i p_j. \tag{30}$$

With $\xi(\vec{q},t)\equiv\beta(t)$ and $\eta_i(\vec{q},t)$ given by Eq. (21), the generators for the symmetry transformations and their first extensions yielding the Noether invariants (27) and (30) for the class of Hamiltonian systems (18) are given by

$$\boldsymbol{U}=\beta(t)\frac{\partial}{\partial t}+\sum_i\left[\tfrac{1}{2}\dot{\beta}q_i+\sum_j a_{ij}q_j+\psi_i(t)\right]\frac{\partial}{\partial q_i}, \tag{31}$$

$$\boldsymbol{U}'=\boldsymbol{U}+\sum_i\left[\tfrac{1}{2}\ddot{\beta}q_i-\tfrac{1}{2}\dot{\beta}\dot{q}_i+\sum_j a_{ij}\dot{q}_j+\dot{\psi}_i(t)\right]\frac{\partial}{\partial\dot{q}_i}.$$

Making use of the auxiliary equations (25) and (26) for $\beta(t)$ and the $\psi_i(t)$, respectively, we may directly prove that $\boldsymbol{U}'$ satisfies the Noether requirement (8) to yield a total time derivative of a function $f(\vec{q},t)$ for the general class of Lagrangian systems (13) with the Hamiltonian of Eq. (18),

$$\boldsymbol{U}'_\psi L=\frac{d}{dt}\sum_i\dot{\psi}_i(t)q_i,\quad\beta(t),a_{ij}\equiv 0,$$

$$\boldsymbol{U}'_{\beta}L+\dot{\xi}L=\frac{d}{dt}\ddot{\beta}(t)\sum_i \tfrac{1}{4}q_i^2,\quad \psi_i(t)\equiv 0,$$

in agreement with Eq. (22). In order to verify that the variation δI_β of the Noether invariant (30) indeed vanishes, we may straightforwardly show that

$$\boldsymbol{U}'_{\beta}I_{\beta}=0\Leftrightarrow\beta(t)\ \text{is a solution of Eq. (25).}$$

With respect to the ψ_i-dependent part of $\boldsymbol{U}'$ acting on the invariant I_{ψ_i} of Eq. (29), we find, similarly,

$$\begin{aligned}\boldsymbol{U}'_{\psi_i}I_{\psi_i}&=\left(\psi_i(t)\frac{\partial}{\partial q_i}+\dot{\psi}_i(t)\frac{\partial}{\partial\dot{q}_i}\right)[\dot{\psi}_i(t)q_i-\psi_i(t)\dot{q}_i]\\&=\psi_i(t)\dot{\psi}_i(t)-\dot{\psi}_i(t)\psi_i(t)\equiv 0.\end{aligned}\tag{32}$$

Obviously, the expression $\boldsymbol{U}'_{\psi_i}I_{\psi_i}$ vanishes separately for each index i, as it should be for the n distinct invariants I_{ψ_i}.

B. Lie symmetry analysis

Another approach to the treatment of the symmetries of a dynamical system has been established by Lie [1]. The class of Lie symmetries is defined by those point transformations (1) that leave the equations of motion invariant,

$$\ddot{q}_i+\frac{\partial V(\vec{q},t)}{\partial q_i}=0,\quad i=1,\dots,n.\tag{33}$$

This coupled set of n second-order equations corresponds to the set of $2n$ first-order canonical equations (19). As the equation of motion (33) is of second order, the condition for a vanishing variation reads

$$\boldsymbol{U}''\left(\ddot{q}_i+\frac{\partial V(\vec{q},t)}{\partial q_i}\right)=0,\quad i=1,\dots,n,\tag{34}$$

with $\boldsymbol{U}''$ the second extension (4) of the generator $\boldsymbol{U}$. Physically, a symmetry mapping of Eq. (33) that is associated with a vanishing variation (34) means to transform the equation of motion into the *same* equation of motion in the new coordinate system. Again, we thereby do not map our given physical system into a different one, but isolate the conditions to be imposed on the point mapping (1) in order to sustain the form of the equation of motion. As the particular dynamical system described by Eq. (33) is given in explicit form and does not involve velocity terms, Eq. (34) simplifies to

$$\eta_i''+\boldsymbol{U}\left(\frac{\partial V(\vec{q},t)}{\partial q_i}\right)=0,$$

which reads with $\boldsymbol{U}$ and η_i'' given by Eqs. (2) and (4),

$$\ddot{\eta}_i-2\dot{\xi}\ddot{q}_i-\ddot{\xi}\dot{q}_i+\xi\frac{\partial^2 V}{\partial q_i\partial t}+\sum_{j=1}^{n}\eta_j\frac{\partial^2 V}{\partial q_i\partial q_j}=0.$$

Similar to the Noether symmetry analysis worked out in Sec. III, this condition can only be fulfilled globally for any velocity vector $\dot{\vec{q}}$ if and only if the sets of linear, quadratic, and cubic velocity terms vanish separately. This requirement leads to the following hierarchy of partial differential equations that must be fulfilled for the given potential $V(\vec{q},t)$ by the functions $\xi(\vec{q},t)$ and $\eta_i(\vec{q},t)$ of the generator (2),

$$\sum_j\sum_k\dot{q}_i\dot{q}_j\dot{q}_k\frac{\partial^2\xi}{\partial q_j\partial q_k}=0,\tag{35a}$$

$$\sum_j\left[2\dot{q}_i\dot{q}_j\frac{\partial^2\xi}{\partial q_j\partial t}-\sum_k\dot{q}_j\dot{q}_k\frac{\partial^2\eta_i}{\partial q_j\partial q_k}\right]=0,\tag{35b}$$

$$\sum_j\left[2\dot{q}_j\frac{\partial^2\eta_i}{\partial q_j\partial t}+\left(2\dot{q}_j\frac{\partial V}{\partial q_i}+\dot{q}_i\frac{\partial V}{\partial q_j}\right)\frac{\partial\xi}{\partial q_j}\right]-\dot{q}_i\frac{\partial^2\xi}{\partial t^2}=0,\tag{35c}$$

$$\sum_j\left[\eta_j\frac{\partial^2 V}{\partial q_i\partial q_j}-\frac{\partial V}{\partial q_j}\frac{\partial\eta_i}{\partial q_j}\right]+\frac{\partial^2\eta_i}{\partial t^2}+2\frac{\partial V}{\partial q_i}\frac{\partial\xi}{\partial t}+\xi\frac{\partial^2 V}{\partial q_i\partial t}=0.\tag{35d}$$

Regarding Eq. (35a), we infer that all second-order derivatives of $\xi(\vec{q},t)$ with respect to the coordinates q_i must be zero. This means that $\xi(\vec{q},t)$ has the general form

$$\xi(\vec{q},t)=\sum_j\alpha_j(t)\,q_j+\beta_L(t),\tag{36}$$

the $\alpha_j(t)$ and $\beta_L(t)$ denoting yet unknown functions of time only. The derivatives of $\xi(\vec{q},t)$ that are contained in Eq. (35) may now be expressed as

$$\frac{\partial\xi}{\partial q_j}=\alpha_j(t),\quad \frac{\partial\xi}{\partial t}=\sum_j\dot{\alpha}_j(t)q_j+\dot{\beta}_L(t),$$

$$\frac{\partial^2\xi}{\partial q_j\partial t}=\dot{\alpha}_j(t),\quad \frac{\partial^2\xi}{\partial t^2}=\sum_j\ddot{\alpha}_j(t)q_j+\ddot{\beta}_L(t).$$

Equation (35b) is therefore globally fulfilled if

$$\frac{\partial^2\eta_i}{\partial q_j\partial q_k}=\dot{\alpha}_j\delta_{ik}+\dot{\alpha}_k\delta_{ij},\tag{37}$$

the δ_{ik} and δ_{ij} meaning Kronecker symbols. The general form of $\eta_i(\vec{q},t)$ is obtained from a formal integration of Eq. (37), introducing yet undetermined functions of time $\gamma_{ij}(t)$ and $\phi_i(t)$,

$$\eta_i(\vec{q},t)=\sum_j[\dot{\alpha}_j(t)q_iq_j+\gamma_{ij}(t)q_j]+\phi_i(t).\tag{38}$$

The derivatives of $\eta(\vec{q},t)$ following from Eq. (38) are

$$\frac{\partial \eta_i}{\partial q_j}=\dot{\alpha}_j q_i+\gamma_{ij}+\delta_{ij}\sum_k \dot{\alpha}_k q_k ,$$

$$\frac{\partial^2 \eta_i}{\partial q_j \partial t}=\ddot{\alpha}_j q_i+\dot{\gamma}_{ij}+\delta_{ij}\sum_k \ddot{\alpha}_k q_k ,$$

$$\frac{\partial^2 \eta_i}{\partial t^2}=\sum_j \left[\dddot{\alpha}_j q_i q_j+\ddot{\gamma}_{ij} q_j\right]+\ddot{\phi}_i .$$

The conditions the time functions $\alpha_j(t)$, $\beta_L(t)$, $\gamma_{ij}(t)$, and $\phi_i(t)$ must obey in order to yield a valid symmetry transformation (34) are obtained inserting $\xi(\vec{q},t)$ of Eq. (36) and $\eta_i(\vec{q},t)$ from Eq. (38) together with their respective partial derivatives into Eqs. (35c) and (35d). Distinguishing between terms that depend on $\dot{\vec{q}}$ and those that do not, the expression following from Eq. (35c) can be split into two sums that must vanish separately,

$$\sum_j \dot{q}_j(2\dot{\gamma}_{ij}-\ddot{\beta}_L\delta_{ij})=0, \tag{39a}$$

$$\sum_j \left[\ddot{\alpha}_j(2\dot{q}_j q_i+\dot{q}_i q_j)+\alpha_j\left(2\dot{q}_j\frac{\partial V}{\partial q_i}+\dot{q}_i\frac{\partial V}{\partial q_j}\right)\right]=0. \tag{39b}$$

Equation (39a) can only be fulfilled globally if $2\dot{\gamma}_{ij}=\ddot{\beta}_L\delta_{ij}$ for all indices i and j. This means after time integration

$$\gamma_{ij}(t)=\tfrac{1}{2}\dot{\beta}_L(t)\delta_{ij}+b_{ij}, \tag{40}$$

with b_{ij} denoting the integration constants. With this result, $\eta_i(\vec{q},t)$ of Eq. (38) may be rewritten as

$$\eta_i(\vec{q},t)=\sum_j \dot{\alpha}_j(t)q_i q_j+\sum_j \left[\tfrac{1}{2}\dot{\beta}_L(t)\delta_{ij}+b_{ij}\right]q_j+\phi_i(t). \tag{41}$$

The $\dot{\vec{q}}$-dependent terms of Eq. (35c) account for Eq. (39b). Due to the coupling of the degrees of freedom that is induced by the potential $V(\vec{q},t)$, it represents a set of n auxiliary equations for the n functions of time $\alpha_1(t),\ldots,\alpha_n(t)$. Apart from particular potentials $V(\vec{q},t)$, this set may be solved only along the system path $\vec{q}(t),\dot{\vec{q}}(t)$ that emerges as the solution of the n equations of motion (33).

Finally, the terms of the hierarchy (35) that do not depend on $\dot{\vec{q}}$ must satisfy Eq. (35d). Replacing γ_{ij} according to Eq. (40), we get three independent differential equations for the time functions $\beta_L(t)$, $\phi_i(t)$, and $\alpha_i(t)$,

$$\dddot{\beta}_L q_i+\dot{\beta}_L\left[3\frac{\partial V}{\partial q_i}+\sum_j q_j\frac{\partial^2 V}{\partial q_i\partial q_j}\right]+2\beta_L\frac{\partial^2 V}{\partial q_i\partial t}-2\sum_j\left[b_{ij}\frac{\partial V}{\partial q_j}-\frac{\partial^2 V}{\partial q_i\partial q_j}\sum_k b_{jk}q_k\right]=0, \tag{42a}$$

$$\ddot{\phi}_i(t)+\sum_j \phi_j(t)\frac{\partial^2 V}{\partial q_i\partial q_j}=0, \tag{42b}$$

$$\sum_j\left[\dddot{\alpha}_j q_i q_j+\dot{\alpha}_j\left(q_j\frac{\partial V}{\partial q_i}-q_i\frac{\partial V}{\partial q_j}\right)+q_j\frac{\partial^2 V}{\partial q_i\partial q_j}\sum_k \dot{\alpha}_k q_k+\alpha_j q_j\frac{\partial^2 V}{\partial q_i\partial t}\right]=0. \tag{42c}$$

As the degrees of freedom are coupled by the potential, each equation stands for a set of n coupled equations, with the index ranging from $i=1,\ldots,n$. In terms of the solutions of the set of differential equations (39b) and (42), the generator $\boldsymbol{U}_L$ of the symmetry transformation (34) is given by

$$\boldsymbol{U}_L=\left[\beta_L(t)+\sum_i \alpha_i(t)q_i\right]\frac{\partial}{\partial t}+\sum_i\left[\tfrac{1}{2}\dot{\beta}_L q_i+\sum_j b_{ij}q_j+\phi_i(t)+q_i\sum_j \dot{\alpha}_j q_j\right]\frac{\partial}{\partial q_i}. \tag{43}$$

Obviously, this operator formally agrees for $\alpha_i\equiv 0$ with the generator (31) of the Noether symmetry transformation treated in Sec. V. Nevertheless, we must keep in mind that the coefficients of the operators (31) and (43) are different in general as they follow from a different set of auxiliary equations. Their interrelation becomes transparent considering that Eqs. (42a) and (42b) are partial q_i derivatives of the respective equations (25) and (26) of the Noether symmetry analysis. Thus, Eqs. (42a) and (42b) can be formally written as the partial q_i derivative of the Noether condition (8),

$$\frac{\partial}{\partial q_i}\left[\boldsymbol{U}_L' L+\dot{\beta}_L(t)L-\frac{df_L(\vec{q},t)}{dt}\right]=0, \tag{44}$$

the operator $\boldsymbol{U}_L'$ given by the first extension of Eq. (43), the Lagrangian L by Eqs. (13) and (18), and the particular gauge function f_L for the actual system that corresponds to Eq. (22) given by

$$f_L(\vec{q},t)=\ddot{\beta}_L(t)\sum_{i=1}^n \tfrac{1}{4}q_i^2+\sum_{i=1}^n \dot{\phi}_i(t)q_i .$$

Regarding the homogeneous equations for the α_i, we observe that Eqs. (39b) and (42c) impose a set of $2n$ conditions for the n functions of time $\alpha_i(t)$. We conclude that—apart from very specific potentials $V(\vec{q},t)$—these conditions cannot be satisfied. This means that in most cases Eqs. (39b) and (42c) admit the trivial solution $\vec{\alpha}(t)\equiv 0$ only, hence no $\vec{\alpha}$-related Lie symmetries exist. The one-dimensional time-dependent harmonic-oscillator system is one exception. It is easily shown that Eqs. (39b) and (42c) are compatible for this particular system, leading to the well-known additional Lie symmetries [11] that exist in addition to the Noether symmetries. In Sec. VI, we demonstrate that these equations also admit a nontrivial solution for the Kepler system—yielding a yet unreported Lie symmetry of this system. Kepler's third law is shown to originate from a particular solution of Eq. (42a) for $\beta_L(t)$. We will furthermore show that the familiar invariants given by the energy conservation law, the conservation of the angular momentum, and the Runge-Lenz vector are Noether symmetries. Finally, two new Noether invariants for the Kepler system are derived from the solutions of Eq. (25) with $\beta(t)\neq\text{const}$.

VI. EXAMPLE: KEPLER SYSTEM

A. Equation of motion

The classical Kepler system is a two-body problem with the mutual interaction following an inverse square force law. In the frame of the reference body, the Cartesian coordinates q_1,q_2 of its counterpart may be described in the plane of motion by

$$\ddot{q}_i+\mu(t)\frac{q_i}{\sqrt{(q_1^2+q_2^2)^3}}=0,\quad i=1,2, \tag{45}$$

with $\mu(t)=G[m_1(t)+m_2(t)]$ the time-dependent gravitational coupling strength that is induced by time-dependent masses $m_1(t)$ and $m_2(t)$ of the interacting bodies. We may regard the equation of motion (45) to originate from the Hamiltonian

$$H(\vec{q},\vec{p},t)=\tfrac{1}{2}p_1^2+\tfrac{1}{2}p_2^2+V(\vec{q},t) \tag{46}$$

containing the interaction potential

$$V(\vec{q},t)=-\frac{\mu(t)}{\sqrt{q_1^2+q_2^2}}=-\frac{\mu(t)}{r}. \tag{47}$$

B. Noether symmetry analysis

The complete set of Noether invariants (29) and (30) together with its related generator (31) is obtained by finding the complete set of solutions of the differential equations (25) and (28) for the particular potential (47).

1. Solutions related to $\beta(t)$ and a_{ij}

We start with the inhomogeneous part of Eq. (25) originating from a nonvanishing antisymmetric matrix (a_{ij}) that is contained in the general solution of Eq. (20b). For our actual two-dimensional system, this matrix cannot contain more than one independent element, viz. $a_{11}=a_{22}=0$, $a_{12}=-a_{21}$. The double sum of Eq. (25) thus reads, explicitly,

$$\sum_{i=1}^{2}\sum_{j=1}^{2}a_{ij}q_j\frac{\partial V}{\partial q_i}=a_{12}q_2\frac{\mu q_1}{r^3}+a_{21}q_1\frac{\mu q_2}{r^3}$$

$$=\frac{\mu}{r^3}a_{12}(q_1q_2-q_2q_1)\equiv 0$$

for arbitrary constants a_{12}. Therefore, the auxiliary equation (25) has the nontrivial solution $a_{12}\neq 0$, independently of $\beta(t)$. Defining $a_{12}=1$, we thus obtain the separate invariant I_a from Eq. (30),

$$I_a=q_1p_2-q_2p_1. \tag{48}$$

Obviously, this invariant represents Kepler's second law, stating that the angular momentum is a conserved quantity. The associated generator $\boldsymbol{U}_a$ of the symmetry transformation is readily obtained from Eq. (31) for $a_{12}=-a_{21}=1$,

$$\boldsymbol{U}_a=q_2\frac{\partial}{\partial q_1}-q_1\frac{\partial}{\partial q_2}. \tag{49}$$

The homogeneous part of Eq. (25) forms a separate auxiliary equation for $\beta(t)$. For our given potential of Eq. (47), we find the third-order equation

$$\dddot{\beta}(t)-\dot{\beta}(t)\frac{2\mu(t)}{r^3(t)}-\beta(t)\frac{4\dot{\mu}(t)}{r^3(t)}=0. \tag{50}$$

With the Hamiltonian (46), and $\beta(t)$ a solution of Eq. (50), the associated invariant I_β is given by

$$I_\beta=\beta(t)H-\tfrac{1}{2}\dot{\beta}(t)(q_1p_1+q_2p_2)+\tfrac{1}{4}\ddot{\beta}(t)(q_1^2+q_2^2). \tag{51}$$

The generators of the symmetry transformations pertaining to the three linear independent solutions of Eq. (50) follow from Eq. (31) as

$$\boldsymbol{U}_{\beta_i}=\beta_i(t)\frac{\partial}{\partial t}+\tfrac{1}{2}\dot{\beta}_i(t)\left(q_1\frac{\partial}{\partial q_1}+q_2\frac{\partial}{\partial q_2}\right). \tag{52}$$

For the conventional case of a *constant* coupling strength $[\dot{\mu}(t)=0]$, the auxiliary equation (50) has the particular solution $\beta_1(t)=1$. With this solution, the invariant (51) reduces to

$$I_{\beta_1=1}=H,$$

which provides the familiar result that the instantaneous system energy H that is given by the Hamiltonian (46) is a conserved quantity if H does not depend on time explicitly. The generator of the corresponding symmetry transformation then simplifies to

$$\boldsymbol{U}_{\beta_1=1}=\frac{\partial}{\partial t}. \tag{53}$$

As for all time-independent systems, two nonconstant solutions $\beta_{2,3}(t)$ of Eq. (50) always exist. For the Kepler system with $\dot{\mu}=0$, these solutions can be expressed as

$$\beta_{2,3}(t)=\int_{t_0}^{t}\frac{u(\theta(\tau))d\tau}{1+\varepsilon\cos\theta(\tau)},$$

where the function $u(\theta)$ is one of the two solutions of the differential equation,

$$\frac{d^2u}{d\theta^2}+\left(1-\frac{3}{1+\varepsilon\cos\theta}\right)u(\theta)=0,$$

and $\theta(t)$ the polar angle of the elliptical trajectory with eccentricity ε. These independent solutions $\beta_{2,3}(t)$ induce two additional nonlocal invariants $I_{\beta_{2,3}}$ of the form of Eq. (51) which—to the authors' knowledge—have not been previously reported.

2. Solutions related to $\psi_i(t)$

The ψ_i-related invariants (29) are obtained from the solutions of the auxiliary equations (28). Inserting our given equation of motion (45) into Eq. (28), we find

$$\ddot{\psi}_i(t)+\psi_i(t)\frac{\mu(t)}{r^3(t)}=0. \tag{54}$$

With $\psi_i(t)$ and $\dot{\psi}_i(t)$ a solution of the auxiliary equation (54), the associated Noether invariants (29) read

$$I_{\psi_i}=\dot{\psi}_i(t)q_i-\psi_i(t)\dot{q}_i,\quad i=1,2. \tag{55}$$

The two independent generators of the symmetry transformation that result from the two linear independent solutions of Eq. (54) are

$$\boldsymbol{U}_{\psi_i}=\psi_i(t)\frac{\partial}{\partial q_i},\quad i=1,2. \tag{56}$$

It is again instructive to contemplate in particular the time-independent case. We may easily convince ourselves by direct insertion that

$$\psi_i(t)=q_1(t)\dot{q}_1(t)+q_2(t)\dot{q}_2(t) \tag{57}$$

is a solution of Eq. (54) provided that $\dot{\mu}(t)=0$. Inserting Eq. (57) and its total time derivative

$$\dot{\psi}_i(t)=\dot{q}_1^2(t)+\dot{q}_2^2(t)-\frac{\mu}{r(t)}$$

into Eq. (55), the invariants read, explicitly,

$$I_{\psi_1}=q_1\dot{q}_2^2-q_2\dot{q}_1\dot{q}_2-q_1\frac{\mu}{r}, \tag{58a}$$

$$I_{\psi_2}=q_2\dot{q}_1^2-q_1\dot{q}_1\dot{q}_2-q_2\frac{\mu}{r}. \tag{58b}$$

Obviously, the Noether invariants (58) represent the two components of the Runge-Lenz vector. This result contrasts with the usual perception of the Runge-Lenz vector as a "non-Noether invariant" [12]. Nevertheless, we must be very careful writing the Noether invariants (58) in this form. The requirement $\boldsymbol{U}'_{\psi_i}I_{\psi_i}=0$ of Eq. (32) for the first extension of the generator (56) acting on the invariant (55) is satisfied if and only if the invariant is written in the form of Eq. (55) with $\psi_i(t)$ given by Eq. (57). Only in this form is the right distinction between spatial and time dependence made in Eq. (55)—with $\psi_i(t)$ written as a function of time only that is defined along the solution path $(\vec{q}(t),\dot{\vec{q}}(t))$ of the equations of motion.

C. Lie symmetry analysis

Similar to the Noether analysis, we may systematically isolate the complete set of Lie symmetries of the equation of motion (45) by finding all solutions of the auxiliary equations (39b) and (42) for the coefficients $\vec{\alpha}(t)$, $\beta_L(t)$, $\vec{\phi}(t)$, and the constant matrix (b_{ij}) that constitute the Lie generator (43).

1. Solutions related to $\alpha_i(t)$

We start our Lie analysis with the time functions $\alpha_i(t)$, given as the simultaneous solutions of Eqs. (39b) and (42c). With Eq. (47) the potential of the Kepler system, the condition (39b) takes on the particular form

$$\ddot{\alpha}_i(t)+\alpha_i(t)\frac{\mu}{r^3}=0,\quad i=1,2. \tag{59}$$

The total time derivative of Eq. (59) inserted into Eq. (42c) then provides the condition for Eqs. (39b) and (42c) to be simultaneously satisfied,

$$(\dot{\alpha}_1q_1+\dot{\alpha}_2q_2)(q_1^2+q_2^2)=(\alpha_1q_1+\alpha_2q_2)(q_1\dot{q}_1+q_2\dot{q}_2).$$

The obvious solution is to identify the time functions $\alpha_i(t)$ with the time evolution of the coordinates $q_i(t)$,

$$\alpha_i(t)=cq_i(t),\quad i=1,2.$$

The related generator of this Lie symmetry reads

$$\boldsymbol{U}_{L,\alpha}=[\alpha_1(t)q_1+\alpha_2(t)q_2]\frac{\partial}{\partial t}+[\dot{\alpha}_1(t)q_1+\dot{\alpha}_2(t)q_2]\times\left(q_1\frac{\partial}{\partial q_1}+q_2\frac{\partial}{\partial q_2}\right). \tag{60}$$

We note that the identification of the time evolution of $\alpha_i(t)$ with the time evolution of the spatial coordinates $q_i(t)$ holds for *arbitrary* time evolutions $\mu(t)$ of the coupling strength. The question whether a physical interpretation of this yet unreported Lie symmetry of the Kepler system exists must be left unanswered at this point. However, an interesting connection between the Lie symmetry generator (60) and the Noether invariants of Eqs. (48) and (51) is revealed by letting the first extension of the operator (60) act on I_a and I_β, respectively. Provided that $\beta(t)$ is a solution of Eq. (50), we find

$$\boldsymbol{U}'_{L,\alpha} I_a = 0, \quad \boldsymbol{U}'_{L,\alpha} I_\beta = 0.$$

Obviously, the Noether invariants I_a and I_β are also invariants with respect to the Lie symmetry that is generated by Eq. (60). This shows that a unique correlation of invariants and symmetry operators is not possible.

2. Solutions related to $\boldsymbol{\beta_L(t)}$ and $\boldsymbol{b_{ij}}$

In the next step, we work out the set of solutions $\beta_L(t)$ of Eq. (42a) for the potential (47). In our particular case, we encounter the same condition for both indices $i=1,2$, viz.,

$$\dddot{\beta}_L + \dot{\beta}_L \frac{\mu(t)}{r^3} + \beta_L \frac{2\dot{\mu}(t)}{r^3} - \frac{6\mu(t)}{r^5}[b_{11}q_1^2 + (b_{12}+b_{21})q_1 q_2 + b_{22}q_2^2] = 0. \tag{61}$$

We may easily identify a particular solution of this inhomogeneous differential equation. For $\beta_L(t)=0$, Eq. (61) is identically satisfied for any $\vec{q}(t)$ if $b_{11}=b_{22}=0$ and $b_{12}=-b_{21}\neq 0$. From this nontrivial solution, we get the following contribution to the generator (43):

$$\boldsymbol{U}_{L,b} = q_2 \frac{\partial}{\partial q_1} - q_1 \frac{\partial}{\partial q_2},$$

which agrees which the Noether operator (49) that represents the conservation law of the angular momentum (48).

Each fundamental solution $\beta_{L,i}(t)$, $i=1,2,3$ of the homogeneous part of third-order Eq. (61),

$$\dddot{\beta}_L + \dot{\beta}_L \frac{\mu(t)}{r^3} + \beta_L \frac{2\dot{\mu}(t)}{r^3} = 0, \tag{62}$$

is associated with the generator

$$\boldsymbol{U}_{L,\beta_{L,i}} = \beta_{L,i}(t)\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{\beta}_{L,i}(t)\left(q_1\frac{\partial}{\partial q_1} + q_2\frac{\partial}{\partial q_2}\right).$$

This operator formally agrees with the corresponding operator (52) of the Noether symmetry analysis. Yet, the respective coefficients $\beta_i(t)$ and $\beta_{L,i}(t)$ are *different* in general as they emerge as solutions from different auxiliary equations (50) and (62).

For the time-independent system [$\mu(t)=\mathrm{const}$], the function $\beta_{L,1}(t)=1$ is a particular solution of Eq. (61). The generator $\boldsymbol{U}_{L,\beta_L=1}$,

$$\boldsymbol{U}_{L,\beta_L=1} = \frac{\partial}{\partial t},$$

agrees with the Noether operator (53) representing the energy conservation law.

A second particular solution of Eq. (61) for $\dot{\mu}(t)=0$ is easily shown to exist for $\beta_L(t)=t$. With this solution, Eq. (61) is fulfilled identically for $b_{12}=b_{21}=0$ and $b_{11}=b_{22}=\frac{1}{6}$. The related contribution to the generator (43) reads

$$\boldsymbol{U}_{L,\beta_L=t} = t\frac{\partial}{\partial t} + \tfrac{2}{3}q_1\frac{\partial}{\partial q_1} + \tfrac{2}{3}q_2\frac{\partial}{\partial q_2}. \tag{63}$$

This generator depends on both the time and the spatial coordinates. It thereby describes a symmetry between the spatial and the time coordinates for this system, which reflects Kepler's third law. We observe that the auxiliary equations of the Noether analysis of Sec. VI B do not admit a solution leading to the generator (63). Therefore, the symmetry generated by Eq. (63) is referred to as a "non-Noether symmetry" [12]. Prince and Eliezer also showed that the Runge-Lenz vector (58) can be derived on the basis of the non-Noether symmetry generated by Eq. (63). This means that the variation of I_{ψ_i} vanishes under the action of this symmetry transformation,

$$\boldsymbol{U}'_{L,\beta_L=t} I_{\psi_i} = 0,$$

hence that the Runge-Lenz vector constitutes an invariant with respect to this non-Noether symmetry. On the other hand, we have shown that the Runge-Lenz vector in the representation of Eq. (55) with $\psi_i(t)$ given by Eq. (57) also embodies a Noether invariant. We conclude that the classification of the Runge-Lenz vector as a "non-Noether invariant" is not justified. This indicates once more that it is not possible to uniquely attribute an invariant to a symmetry operator.

For $\mu(t)=\mathrm{const}$, we can express the remaining two independent solutions of the homogeneous part of Eq. (61) in explicit form. A comparison of Eq. (62) with the equation of motion (45) yields

$$\dot{\beta}_{L,i}(t) = c_i q_i(t), \quad i=2,3,$$

hence to the integral representation of $\beta_{L,i}(t)$,

$$\beta_{L,i}(t) = c_i \int_{t_0}^{t} q_i(\tau)\, d\tau, \quad i=2,3, \tag{64}$$

in agreement with an approach that has been worked out earlier by Krause [13].

3. Solutions related to $\boldsymbol{\phi_i(t)}$

Finally, we work out the symmetries that are related to the solutions of auxiliary equation (42b). For the potential (47) and $i=1,2$, this equation has the particular representation

$$\ddot{\phi}_i(t) - \frac{\mu(t)}{r^5}\{3q_i[\phi_1(t)q_1 + \phi_2(t)\,q_2] - \phi_i(t)(q_1^2+q_2^2)\} = 0. \tag{65}$$

With $\phi_1(t)$ and $\phi_2(t)$, two linear independent solutions of the coupled set of second-order equations (65), the related part of the generator (43) reads

$$\boldsymbol{U}_{L,\phi} = \phi_1(t)\frac{\partial}{\partial q_1} + \phi_2(t)\frac{\partial}{\partial q_2}.$$

Again, the auxiliary equations (65) for the coefficients $\phi_i(t)$ differ from the respective equations (54) for the coefficients of its Noether counterpart (56). As the subsequent solutions are different in general, we obtain different time evolutions of the Noether and Lie symmetries.

In the case of the autonomous system with $\dot{\mu}(t)=0$, the auxiliary equation (65) possesses a simple general solution. Comparing Eq. (65) with the time derivative of the equation of motion (45),

$$\dddot{q}_i(t) - \frac{\mu}{r^5}[3q_i(\dot{q}_1 q_1 + \dot{q}_2 q_2) - \dot{q}_i(q_1^2 + q_2^2)] = 0,$$

we immediately see that

$$\phi_i(t) = c\dot{q}_i(t), \quad i=1,2 \tag{66}$$

is a solution of Eq. (65). Herein, c denotes an arbitrary constant. For the time-independent Kepler system, the functions of time $\phi_1(t)$ and $\phi_2(t)$ thus coincide up to a constant factor with the time evolution of the velocities $\dot{q}_1(t)$ and $\dot{q}_2(t)$. We finally note the interesting result that the variation of the Runge-Lenz vector (58) vanishes as well under the action of the Lie symmetry generator $\boldsymbol{U}_{L,\phi}$,

$$\boldsymbol{U}'_{L,\phi} I_{\psi_i} = 0,$$

which again confirms our observation that a unique correlation between invariant and symmetry does not exist.

D. Discussion

The characteristic functions of time that are contained in both the Noether as well as the Lie symmetry generators reflect the specific symmetry properties of the dynamical system in question. The Noether symmetry generator (31) is determined by the set of time functions $\beta(t)$, $\vec{\psi}(t)$, and constants (a_{ij}) that follow as solutions from differential equations (25) and (28). Similarly, the time functions $\vec{\alpha}(t)$, $\beta_L(t)$, $\vec{\phi}(t)$, and constants (b_{ij})—following from the auxiliary equations (39b) and (42)—constitute the Lie symmetry generator (43). The auxiliary equations for the time-dependent coefficients of the symmetry generators generally depend on $\vec{q}$, the system's potential $V(\vec{q},t)$, and on the partial derivatives of this potential. Nevertheless, particular solutions of these auxiliary equations may exist that are decoupled from the solutions $\vec{q}(t)$ of the equations of motion. Then, the underlying symmetry reflects a fundamental property of the dynamical system. As examples, we quote the solution $\beta(t)=1$ of Eq. (25) that exists for all cases where the Hamiltonian (18)—hence the potential $V(\vec{q},t)$—does not depend on time explicitly. The corresponding invariant $I_\beta = H$ represents the energy conservation law. Similarly, for the Kepler system the solution $a_{12}=1$ of the inhomogeneous part of Eq. (25) provides the fundamental law for the conservation of the angular momentum. Furthermore, the solution $\beta_L(t)=t$ of Eq. (61) for $\dot{\mu}(t)=0$ reflects the symmetry that is associated with Kepler's third law.

Nevertheless, apart from these important particular solutions of the sets of auxiliary equations, a wide variety of solutions exists in addition that explicitly depends on $\vec{q}(t)$. For instance, in the case of a Hamiltonian system whose potential $V(\vec{q},t)$ does depend on time explicitly, the solution $\beta(t)=1$ of Eq. (25) does not exist. Apart from particular cases associated with quadratic potentials, the solution $\beta(t)$ then depends on the particular time evolution of $\vec{q}(t)$, and hence on the time evolution of the potential $V(\vec{q}(t),t)$ and its partial derivatives. As the time function $\vec{q}(t)$ is given by the solution of the equation of motion (19), the solution $\beta(t)$ can only be found by integrating Eq. (25) *simultaneously* with the equations of motion (19). This, in turn, means that the existence of the invariant cannot help us in any way to ease the problem of solving the equations of motion by reducing the system's order. However, the related invariant exists and reflects a particular symmetry of the system's time evolution. The functional structure of the respective auxiliary equation is well defined as it represents—in conjunction with the set of equations of motion—a closed equation with the time t the only independent variable. The dependence of a solution $\beta(t)$ on the time evolution of $\vec{q}(t)$ does *not* indicate that the auxiliary equations are functions of both $\vec{q}$ and t. By virtue of their definition as integration "constants," all coefficients of the auxiliary equations must represent functions of time only. We must therefore regard $\vec{q}=\vec{q}(t)$ as time-dependent coefficients of the auxiliary equations—defined along the system trajectory $\vec{q}(t)$ that emerges as the solution of the equations of motion. This requirement appears natural looking back on Noether's theorem in the form of Eq. (10). It indicates that the expression in brackets constitutes an invariant if and only if the Euler-Lagrange terms of the second sum vanish. This is exactly the case along the system trajectory, defined as the solution of the Euler-Lagrange equations.

In our example of the time-independent Kepler system, we present an explicit solution (57) of the auxiliary equation (54). The functions $\psi_i(t)$ are understood as functions of time satisfying Eq. (65). Of course, we are free to identify the time evolution of the $\psi_i(t)$ with the *time evolution* of the particle coordinates $q_i(t)$ and $\dot{q}_i(t)$, or combinations thereof. With this understanding, the functions $\psi_i(t)$ remain functions of time only—which is crucial for Eq. (32) to be satisfied. The Runge-Lenz vector of the Kepler system can then be conceived as a Noether invariant, generated by a particular operator of the type $\boldsymbol{U}_\psi$.

VII. CONCLUSIONS

We have worked out in detail the Noether and Lie symmetry analyses for a Hamiltonian comprising the explicitly time-dependent general potential $V(\vec{q},t)$. In this general form, the analyses resulted in specific sets of ordinary differential equations for the coefficients of the symmetry generators. The search for Noether and Lie symmetries could thus be reduced to the pursuit of the complete variety of solutions of the sets of auxiliary equations.

The auxiliary equations of the Lie approach were found to agree with the partial q_i derivative of the corresponding Noether equations. For one, this indicates the close relationship between these two approaches. On the other hand, the sets of auxiliary equations are obviously different, hence, apart from particular isotropic systems associated with quadratic potentials, the solution functions are different in general. This means that the time evolutions of Noether and Lie symmetries generally do not agree, hence that the set of Noether symmetries cannot be regarded as a subset of the Lie symmetries.

For the Noether approach, we have seen that there exists a one-to-one correspondence between symmetry and a related invariant. The reason for this is that the Noether auxiliary equations—in conjunction with the equations of motion—can always be cast into the form of a total time derivative. This does not hold for the auxiliary equations that emerge from the Lie symmetry analysis. Therefore, the Lie symmetries are not necessarily associated with closed-form expressions of conserved quantities.

Depending on their specific form for a given potential, the Noether auxiliary equations (25) and (28) as well as the corresponding Lie auxiliary equations (39b) and (42) may have explicit solutions, which then reflect fundamental symmetries of the dynamical system in question. Particular solutions of these equations may exist as well that decouple from the system trajectory $\vec{q}(t)$ representing the solution of the equations of motion as the system moves forward in time. On the other hand, additional solutions of the auxiliary equations exist that explicitly depend on the evolution of $\vec{q}(t)$. These solutions must be taken into consideration as well in order to obtain the full set system symmetries.

With this perception of the auxiliary equations, all invariants of the Kepler system could be derived from Noether's theorem. In particular, the Runge-Lenz vector has been identified as a Noether invariant. In this regard, we confirm the claim made by Sarlet and Cantrijn [14] that "all integrals of Lagrangian systems can indeed be found by a systematic exploration via Noether's theorem." This contrasts with the statement of Prince and Eliezer [12] that "the Runge-Lenz vector evades detection by Noether's theorem." The reason for this discrepancy originates in a different understanding of the coefficients of the auxiliary equations. By virtue of their definition being functions of time only, the coefficients must not depend on $\vec{q}$ and $\dot{\vec{q}}$. Nevertheless, we are free to relate the *time evolution* of these coefficients with the time evolution of $\vec{q}(t)$ and $\dot{\vec{q}}(t)$. With this enhanced understanding of the solutions of the auxiliary equations, a broader solution "spectrum" is obtained, which results in a wider range of invariants emerging within the framework of Noether's theorem. As a consequence, yet unknown nonlocal Noether symmetries of the Kepler system could be isolated.

Furthermore, we have worked out additional solutions of the Lie auxiliary equations for the Kepler system that yield yet unreported symmetries of this system.

We have seen that the variations of the obtained invariants vanish with respect to different Noether as well as Lie symmetry operators. This demonstrates that a unique correspondence between an invariant and a symmetry transformation does not exist. As a consequence, a classification of invariants with respect to a certain symmetry operation is not possible.

Analyzing the symmetries of a given dynamical system, the conditions under which certain solutions of the auxiliary equations cease to exist can be identified as the regions in parameter space where the related symmetries disappear. This way, we may efficiently isolate the causes that render a dynamical system less symmetric, hence more chaotic or even unstable.

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